

Analytic approximation and an improved method for computing the stress-energy of quantized scalar fields in Robertson-Walker Spacetimes

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Abstract

An improved method is given for the computation of the stress-energy tensor of a quantized scalar field using adiabatic regularization. The method works for fields with arbitrary mass and curvature coupling in Robertson-Walker spacetimes and is particularly useful for spacetimes with compact spatial sections. For massless fields it yields an analytic approximation for the stress-energy tensor that is similar in nature to those obtained previously for massless fields in static spacetimes.

Adiabatic regularization is a very useful technique for numerical computations of the stress-energy tensor for quantized fields in cosmological spacetimes. It has been developed for Robertson-Walker, Bianchi Type I, and Gowdy T^3 spacetimes [1–8] and has been used to numerically compute the stress-energy for quantized scalar fields in these spacetimes [4,6,9–14]. It has been proven to be equivalent to point splitting in Robertson-Walker, RW, spacetimes [15,8].

In this paper we present an improved method for computing the stress-energy tensor of a quantized scalar field using adiabatic regularization. The method works for fields with arbitrary mass and curvature coupling in a general RW spacetime. It has two advantages over previous methods. First it is more easily applied to spacetimes with compact spatial sections such as the case of a RW universe with positive spatial curvature. Second it results in an analytic approximation for the stress-energy tensor which can be useful for massless fields and fields with very small masses. The approximation is very similar in nature to those obtained previously for massless fields in static spacetimes [16–18]. It is different from the approximation obtained by summing over all terms in the DeWitt-Schwinger expansion which contain factors of the scalar curvature [19]. By its nature the analytic approximation does not give much information about particle production effects which are inherently non-local. However, it does give information about vacuum polarization effects and can thus give good qualitative and quantitative information about the behavior of the stress-energy tensor when vacuum polarization effects dominate. A further advantage of the analytic approximation is that when one is computing the full renormalized stress-energy tensor for massless fields, the analytic approximation allows one to separate out, at least to some extent, the vacuum polarization part from the particle production part.

In what follows we first discuss our method of computing the stress-energy tensor for quantized scalar fields in RW spacetimes and we derive the analytic approximation. We next discuss the validity and usefulness of the approximation in various cases and finish by comparing it to the full renormalized stress-energy tensor in a particular case in which that tensor is known.

To begin consider a free scalar field with arbitrary mass and curvature coupling in a RW spacetime. The metric for a general RW spacetime can be written ¹

$$ds^2 = a^2(\eta) \left(d\eta^2 - \frac{dr^2}{1 - Kr^2} - r^2 d\Omega^2 \right) . \quad (1)$$

Here $K = 0, +1, -1$ correspond to the cases of zero, positive, and negative spatial curvature respectively. The field ϕ can be expanded in the following manner [20]

$$\phi(x) = \frac{1}{a(\eta)} \int d\tilde{\mu}(k) \left(a_k Y_k(\mathbf{x}) \psi_k(\eta) + a_k^\dagger Y_k^*(\mathbf{x}) \psi_k^*(\eta) \right) \quad (2)$$

with

¹Throughout this paper we use units such that $\hbar = c = 1$. The metric signature is $(+---)$ and the conventions for curvature tensors are $R_{\beta\gamma\delta}^\alpha = \Gamma_{\beta\gamma,\delta}^\alpha - \dots$ and $R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha$.

$$\begin{aligned}
\int d\tilde{\mu}(k) &\equiv \int d^3k \quad K = 0, \\
&\equiv \int_0^\infty dk \sum_{l,m} \quad K = -1 \\
&\equiv \sum_{k,l,m} \quad K = 1 \quad .
\end{aligned} \tag{3}$$

The spatial part of the mode functions $Y_k(\mathbf{x})$ obeys the equation

$$\Delta^{(3)} Y_k(\mathbf{x}) = -(k^2 - K) Y_k(\mathbf{x}) \quad . \tag{4}$$

The time dependent part ψ_k obeys the mode equation

$$\psi_k'' + (k^2 + m^2 a^2 + (\xi - 1/6) a^2 R) \psi_k = 0 \quad . \tag{5}$$

Here m is the mass of the field and ξ its coupling to the scalar curvature R . In spacetimes with the metric (1)

$$R = 6 \left(\frac{a''}{a^3} + \frac{K}{a^2} \right) \quad . \tag{6}$$

The unrenormalized stress-energy tensor is given by the expressions [7,8]

$$\begin{aligned}
\langle 0|T_0^0|0 \rangle_u &= \frac{1}{4\pi^2 a^4} \int d\mu(k) \left(|\psi_k'|^2 + (k^2 + m^2 a^2) |\psi_k|^2 \right. \\
&\quad \left. + 6 \left(\xi - \frac{1}{6} \right) \left[\frac{a'}{a} (\psi_k \psi_k^{*'} + \psi_k^* \psi_k') - \left(\frac{a'^2}{a^2} - K \right) |\psi_k|^2 \right] \right) \tag{7a}
\end{aligned}$$

$$\begin{aligned}
\langle 0|T|0 \rangle_u &= \frac{1}{2\pi^2 a^4} \int d\mu(k) \left(m^2 a^2 |\psi_k|^2 + 6 \left(\xi - \frac{1}{6} \right) \left[|\psi_k'|^2 - \frac{a'}{a} (\psi_k \psi_k^{*'} + \psi_k^* \psi_k') \right. \right. \\
&\quad \left. \left. - \left(k^2 + m^2 a^2 + \frac{a''}{a} - \frac{a'^2}{a^2} + \left(\xi - \frac{1}{6} \right) a^2 R \right) |\psi_k|^2 \right] \right) \tag{7b}
\end{aligned}$$

with

$$\begin{aligned}
\int d\mu(k) &\equiv \int_0^\infty dk k^2 \quad K = 0, -1 \\
&\equiv \sum_{k=1}^\infty k^2 \quad K = 1 \quad .
\end{aligned}$$

To renormalize one subtracts off the renormalization counterterms which come from a WKB expansion of the mode equation. This is the usual method of adiabatic regularization. Schematically one has

$$\langle T_{\mu\nu} \rangle_r = \langle T_{\mu\nu} \rangle_u - \langle T_{\mu\nu} \rangle_{ad} \quad . \tag{8}$$

These counterterms are given in Ref. [7,8]. As discussed in detail in Ref. [8] the adiabatic counterterms in the case $K = 1$ consist of an integral rather than a sum over k . The reason is that the counterterms should be local and thus should be independent of whether the spatial

sections are compact or not. This argument would also apply to $K = 0, -1$ RW spacetimes with periodically identified spatial sections. As a result there is an added difficulty in the case of compact spatial sections in subtracting off the renormalization counterterms.

To improve on the method of adiabatic regularization we expand the renormalization counterterms in inverse powers of k keeping only terms which are ultraviolet divergent. For the case of compact spatial sections the integral is also changed into a sum. We call the resulting expressions $\langle T_{\mu\nu} \rangle_d$. In a general RW spacetime they have the form

$$\begin{aligned} \langle T_0^0 \rangle_d &= \frac{1}{4\pi^2 a^4} \int d\mu(k) \left(k + \frac{1}{k} \left[\frac{m^2 a^2}{2} - 3 \left(\xi - \frac{1}{6} \right) \left(\frac{a'^2}{a^2} - K \right) \right] \right) \\ &\quad + \frac{1}{4\pi^2 a^4} \int d\bar{\mu}(k) \frac{1}{k^3} \left(-\frac{m^4 a^4}{8} - \frac{3m^2 a^2}{2} \left(\frac{a'^2}{a^2} + K \right) + \left(\xi - \frac{1}{6} \right)^2 {}^{(1)}H_0^0 \frac{a^4}{4} \right) \end{aligned} \quad (9a)$$

$$\begin{aligned} \langle T \rangle_d &= \frac{1}{4\pi^2 a^4} \int d\mu(k) \frac{1}{k} \left(m^2 a^2 - 6 \left(\xi - \frac{1}{6} \right) \left(\frac{a''}{a} - \frac{a'^2}{a^2} \right) \right) \\ &\quad + \frac{1}{4\pi^2 a^4} \int d\bar{\mu}(k) \frac{1}{k^3} \left(-\frac{m^4 a^4}{2} - \left(\xi - \frac{1}{6} \right) 3m^2 a^2 \left(\frac{a''}{a} + K \right) \right. \\ &\quad \left. + \left(\xi - \frac{1}{6} \right)^2 {}^{(1)}H_\mu^\mu \frac{a^4}{4} \right) \end{aligned} \quad (9b)$$

with

$$\begin{aligned} \int d\bar{\mu}(k) &\equiv \int_\lambda^\infty dk k^2 \quad K = 0, -1 \\ &\equiv \sum_{k=1}^\infty k^2 \quad K = 1 \quad . \end{aligned}$$

Here λ is an arbitrary lower limit cutoff and

$${}^{(1)}H_{\mu\nu} = 2R_{;\mu\nu} - 2g_{\mu\nu} \square R - \frac{1}{2}g_{\mu\nu} R^2 + 2RR_{\mu\nu} \quad . \quad (10a)$$

In a RW spacetime it has the components

$${}^{(1)}H_0^0 = -\frac{36a'''a'}{a^6} + \frac{72a''a'^2}{a^7} + \frac{18a''^2}{a^6} + \frac{36Ka'^2}{a^6} - \frac{18K^2}{a^4} \quad (10b)$$

$${}^{(1)}H_\mu^\mu = -\frac{36a''''}{a^5} + \frac{144a'''a'}{a^6} - \frac{216a''a'^2}{a^7} + \frac{108a''^2}{a^6} + \frac{72Ka''}{a^5} - \frac{72Ka'^2}{a^6} \quad . \quad (10c)$$

The renormalized stress-energy tensor is then computed by subtracting and adding the quantity $\langle T_\mu^\nu \rangle_d$ to Eq. (8) with the result that

$$\langle T_{\mu\nu} \rangle_r \equiv \langle T_{\mu\nu} \rangle_n + \langle T_{\mu\nu} \rangle_{an} \quad (11a)$$

$$\langle T_{\mu\nu} \rangle_n \equiv \langle T_{\mu\nu} \rangle_u - \langle T_{\mu\nu} \rangle_d \quad (11b)$$

$$\langle T_{\mu\nu} \rangle_{an} \equiv \langle T_{\mu\nu} \rangle_d - \langle T_{\mu\nu} \rangle_{ad} \quad . \quad (11c)$$

In general $\langle T_{\mu\nu} \rangle_n$ must be computed numerically while $\langle T_{\mu\nu} \rangle_{an}$ can always be computed analytically. The result is

$$\begin{aligned}
\langle T_0^0 \rangle_{an} = & \frac{1}{2880\pi^2} \left(-\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 - \frac{3K(K-1)}{a^4} \right) + \frac{m^2}{288\pi^2} G_0^0 \\
& - \frac{m^2 K(K-1)}{192\pi^2 a^2} - \frac{m^4}{64\pi^2} \left[\frac{1}{2} + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right] \\
& + \left(\xi - \frac{1}{6} \right) \left[\frac{{}^{(1)}H_0^0}{288\pi^2} + \frac{K(K-1)}{32\pi^2 a^4} \left(1 + \frac{a'^2}{a^2} \right) \right. \\
& + \frac{m^2}{16\pi^2} G_0^0 \left(3 + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right) + \frac{3Km^2}{8\pi^2 a^2} \Big] \\
& + \left(\xi - \frac{1}{6} \right)^2 \left[\frac{{}^{(1)}H_0^0}{32\pi^2} \left(2 + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right) \right. \\
& \left. \left. - \frac{9}{4\pi^2} \left(\frac{a'^2 a''}{a^7} + \frac{Ka'^2}{a^6} \right) \right] \quad (12a)
\end{aligned}$$

$$\begin{aligned}
\langle T \rangle_{an} = & \frac{1}{2880\pi^2} \left(-\frac{1}{6} {}^{(1)}H_\mu^\mu + {}^{(3)}H_\mu^\mu \right) + \frac{m^2}{288\pi^2} G_\mu^\mu - \frac{m^2 K(K-1)}{96\pi^2 a^2} \\
& - \frac{m^4}{16\pi^2} \left[1 + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right] \\
& + \left(\xi - \frac{1}{6} \right) \left[\frac{{}^{(1)}H_\mu^\mu}{288\pi^2} + \frac{K(K-1)}{16\pi^2 a^4} \left(\frac{a''}{a} - \frac{a'^2}{a^2} \right) \right. \\
& + \frac{m^2}{16\pi^2} G_\mu^\mu \left(3 + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right) \\
& + \frac{3Km^2}{8\pi^2 a^2} - \frac{3m^2 a'^2}{8\pi^2 a^4} \Big] \\
& + \left(\xi - \frac{1}{6} \right)^2 \left[\frac{{}^{(1)}H_\mu^\mu}{32\pi^2} \left(2 + \log \left(\frac{\mu^2 a^2}{4\lambda^2} \right) + \frac{1}{2} K(K+1)(2C + \log \lambda^2) \right) \right. \\
& \left. - \frac{9}{8\pi^2} \left(\frac{4a'''a'}{a^6} - \frac{10a''a'^2}{a^7} + \frac{3a''^2}{a^6} + \frac{4Ka''}{a^5} - \frac{6Ka'^2}{a^6} + \frac{K^2}{a^4} \right) \right] \quad (12b)
\end{aligned}$$

Here $G_{\mu\nu}$ is the Einstein tensor with components

$$G_0^0 = -\frac{3a'^2}{a^4} - \frac{3K}{a^2} \quad (13a)$$

$$G_\mu^\mu = -\frac{6a''}{a^3} - \frac{6K}{a^2} \quad (13b)$$

and ${}^{(3)}H_{\mu\nu}$ is the tensor

$${}^{(3)}H_{\mu\nu} = R_\mu{}^\rho R_{\rho\nu} - \frac{2}{3} R R_{\mu\nu} - \frac{1}{2} R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \frac{1}{4} R^2 g_{\mu\nu} \quad (14a)$$

with components

$${}^{(3)}H_0^0 = \frac{3a'^4}{a^8} + \frac{6Ka'^2}{a^6} + \frac{3K^2}{a^4} \quad (14b)$$

$${}^{(3)}H_\mu^\mu = \frac{12a''a'^2}{a^7} - \frac{12a'^4}{a^8} + \frac{12Ka''}{a^5} - \frac{12Ka'^2}{a^6} \quad (14c)$$

For a massive field $\mu = m$ while for a massless field μ is an arbitrary constant. However, in the massless case the terms containing $\log(\mu^2)$ each have as coefficients multiples of the tensor $^{(1)}H_{\mu\nu}$ which comes from an R^2 term in the gravitational Lagrangian. Thus the terms containing $\log \mu^2$ simply correspond to a finite renormalization of the coefficient of the R^2 term in the gravitational Lagrangian.

Note that if $K = 1$ then $\langle T_{\mu\nu} \rangle_d$ consists of a sum over k while $\langle T_{\mu\nu} \rangle_{ad}$ consists of an integral over k as previously mentioned. Thus either the integral must be converted to a sum or the sum to an integral. We have converted the sum to an integral using the Plana sum formula [21–24]. This formula is

$$\sum_{n=m}^{\infty} f(n) = \frac{1}{2}f(m) + \int_m^{\infty} dx f(x) + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [f(m + it) - f(m - it)] . \quad (15)$$

Because of the way $\langle T_{\mu\nu} \rangle_d$ is defined the third term in the Plana sum formula can be computed exactly. In the traditional form of adiabatic regularization one would convert the integral in the adiabatic counterterms to a sum using the Plana sum formula and then substitute the result into Eq.(8). However, if this is done then, for a massive field, it is not possible to compute the third term in the Plana sum formula analytically. Thus the computation of the renormalized stress-energy tensor is simplified somewhat by our method in the $K = 1$ case. Clearly the same simplification would occur if one was using compact spatial sections for $K = 0$ or $K = -1$ RW spacetimes.

By direct computation one finds that $\langle T_{\mu\nu} \rangle_n$ and $\langle T_{\mu\nu} \rangle_{an}$ are separately conserved. Thus $\langle T_{\mu\nu} \rangle_{an}$ can be used as an analytic approximation for the stress-energy tensor. For $K = 0, -1$ spacetimes it does contain the arbitrary constant λ so it is not unique unless the coefficients of the log terms vanish. However, as can be seen by examining Eqs.(12a) and (12b), changing the value of λ simply corresponds to a finite renormalization of the cosmological constant, R , and R^2 terms in the gravitational Lagrangian. It is important to note that this is only true when using $\langle T_{\mu\nu} \rangle_{an}$ as an analytic approximation. The λ dependent terms do not appear in the full renormalized stress-energy tensor.

Because it depends quartically and quadratically on the mass, $\langle T_{\mu\nu} \rangle_{an}$ is not a good approximation in the large mass limit. Previous numerical work [11] indicates that the relevant condition is likely to be $ma \ll 1$. The quantity $\langle T_{\mu\nu} \rangle_{an}$ is also local in the sense that it depends on the scale factor and its derivatives at a given time η . Thus it cannot accurately describe particle production effects which are inherently nonlocal. However when used as an analytic approximation, it has the potential to do a good job in describing vacuum polarization effects. For the case of the conformally invariant scalar field the analytic approximation is exact. For other cases, it is not usually exact and may not always be quantitatively a good approximation, but qualitatively it can still be very useful. For example the renormalized stress-energy tensor has been computed analytically by Bunch and Davies [25] for a massless minimally coupled scalar field in a $K = 0$ universe undergoing a powerlaw expansion of the form $a = \alpha t^c = \alpha^{1/(1-c)}(1-c)^{c/(1-c)}\eta^{c/(1-c)}$, with $dt = a d\eta$ the proper time. They choose what is effectively the “out” vacuum which means that there is no particle production. Letting $p = c/(1-c)$ we find that the Bunch Davies result can be written in the form

$$\langle T_0^0 \rangle_r = \frac{1}{2880\pi^2} \left[-\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 \right]$$

$$\begin{aligned}
& + \frac{1}{1152\pi^2} {}^{(1)}H_0^0 \left[\log \left(\frac{\mu^2 a^2 \eta^2}{6p(p-1)} \right) - \psi\left(\frac{3}{2} + \nu\right) - \psi\left(\frac{3}{2} - \nu\right) - \frac{4}{3} \right] \\
& + \frac{1}{128\pi^2 a^4 \eta^4} p(p-1)(p^2 + 3p + 4)
\end{aligned} \tag{16a}$$

$$\begin{aligned}
\langle T_\mu^\mu \rangle_r &= \frac{1}{2880\pi^2} \left[-\frac{1}{6} {}^{(1)}H_\mu^\mu + {}^{(3)}H_\mu^\mu \right] \\
& + \frac{1}{1152\pi^2} {}^{(1)}H_\mu^\mu \left[\log \left(\frac{\mu^2 a^2 \eta^2}{6p(p-1)} \right) - \psi\left(\frac{3}{2} + \nu\right) - \psi\left(\frac{3}{2} - \nu\right) - \frac{4}{3} \right] \\
& + \frac{1}{32\pi^2 a^4 \eta^4} p(p-1)(3p^2 + 5p + 4) .
\end{aligned} \tag{16b}$$

Here $\nu \equiv |1 - 3c|/(2|1 - c|)$. The analytic approximation on the other hand gives (if we absorb the infrared cutoff λ into the arbitrary constant μ)

$$\begin{aligned}
\langle T_0^0 \rangle_{an} &= \frac{1}{2880\pi^2} \left[-\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 \right] \\
& + \frac{1}{1152\pi^2} {}^{(1)}H_0^0 \left[\log \left(\frac{\mu^2 a^2}{4} \right) + 2 \right] - \frac{1}{32\pi^2 a^4 \eta^4} p(p-1)(3p^2 + p)
\end{aligned} \tag{16c}$$

$$\begin{aligned}
\langle T_\mu^\mu \rangle_{an} &= \frac{1}{2880\pi^2} \left[-\frac{1}{6} {}^{(1)}H_\mu^\mu + {}^{(3)}H_\mu^\mu \right] \\
& + \frac{1}{1152\pi^2} {}^{(1)}H_\mu^\mu \left[\log \left(\frac{\mu^2 a^2}{4} \right) + 2 \right] + \frac{1}{32\pi^2 a^4 \eta^4} p(p-1)(3p^2 + 15p + 4) .
\end{aligned} \tag{16d}$$

From these expressions one sees that some of the terms in the analytic approximation are identical to those in the exact expression. Those that differ do so only by coefficients which in most cases are of the same order of magnitude. Thus $\langle T_{\mu\nu} \rangle_{an}$ clearly can serve as a useful approximation in this case. Given this fact, one can immediately deduce for example that if $\xi \neq 0$ then the stress-energy tensor will continue to have terms of the form

$$\frac{1}{a^4 \eta^4} (c_1 + c_2 \log(\mu^2 a^2)) . \tag{17}$$

We have presented an improved method to compute the stress-energy tensor for a scalar field in a RW spacetime using adiabatic regularization. The method has a computational advantage over the usual method for spacetimes with compact spatial sections where the unrenormalized terms contain a mode sum and the adiabatic counterterms an integral. Using the method we have derived an analytic approximation for the stress-energy tensor which is particularly useful for massless fields when vacuum polarization effects dominate.

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